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DEVELOPMENT OF SHIP WAVES

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Essentially unstable wave modes (formation and development of ship's wake, its boundaries and zones outside it), as well as the conditions of establishment of ship's waves are investigated. Asymptotic analysis of the problem with respect to four characteristic parameters shows the existence of eleven zones in which solutions are qualitatively different. In eight of these zones solutions have qualitatively new asymptotic properties, while complete asymptotic expansions that take into account unstable contributions are derived in the three previously known zones. All this makes it possible to define the formation, development, and onset of ship's wakes on the fluid free surface at any instant of time.

Previously similar unstable problems were restricted to the investigation of the region of the established ship's wake and its boundary [1, 2]. More detailed investigations of zones with different behavior of the free surface, but in the stabilized case of were described in [3].

1. We consider the problem of motion of a perfect incompressible fluid of infinite depth, induced by the uniform and rectilinear translation over the fluid free surface by the following system of normal stresses concentrated at a point:

$$\frac{\partial \mathbf{V}}{\partial t} = -\frac{1}{\rho}\operatorname{grad} P, \quad \operatorname{div}\mathbf{V} = 0 \tag{1.1}$$

$$P = p_0 + \rho g \zeta, \quad V_z = \frac{\partial \zeta}{\partial t}, \quad z = 0$$

$$\{\mathbf{V}, P\} = 0, \quad z = -\infty; \quad \{\mathbf{V}, P, \zeta\} = 0, \quad x^2 + y^2 = \infty$$

$$\{\mathbf{V}, \zeta, p_0\} = 0, \quad t = 0; \quad \mathbf{V} = \{V_x, V_y, V_z\}$$

$$p_0 = \lim_{\beta \to 0} \frac{q\beta}{2\pi}\left[(x + Ut)^2 + y^2 + \beta^2\right]^{-3/2}, \quad t > 0$$

where x, y, z are orthogonal Cartesian coordinates, $\mathbf{V}$ is the velocity vector, ζ is the rise of the fluid free surface at instant of time t, P is the hydrodynamic pressure, p_0 is the system of external normal stresses, and U the velocity of its motion, ρ is the fluid density, and g the acceleration of gravity.

Applying to problem (1.1) the method of Laplace integral transformation with respect to t and the Fourier transformation with respect to x and y, we obtain for the rise of the fluid free surface the integral representation

$$\zeta = -\frac{qgK}{4\pi^2\rho U^4} \lim_{\beta\to 0} \int_0^T \int_0^{2\pi} \int_0^\infty a^{3/2} \sin(K\tau\sqrt{a})\, e^{-\mu(\theta,\tau) a}\, da\, d\theta\, d\tau \qquad (1.2)$$

$$\mu(\theta,\tau) = \frac{g\beta}{U^2} + iKR(\tau)\cos(\theta-\gamma),$$

$$R(\tau) = [(\tau - \cos\gamma)^2 + \sin^2\gamma]^{1/2}$$

$$K = \frac{gr}{U^2}, \quad T = \frac{Ut}{r} \quad (x + Ut = r\cos\gamma, y = r\sin\gamma)$$

where r and γ are coordinates of a point in the movable polar coordinate system connected to the stationary input system by the formulas appearing in parentheses.

2. Let us analyze ζ in the region of the forward wave front ($KT^2 \leqslant \text{const} < \infty$, $T \leqslant \text{const} < \infty$) and, also, for limited times near the perturbation source ($KT \leqslant \text{const} < \infty$, $T \leqslant \text{const} < \infty$), but excluding the coordinate origin. For this we expand in formula (1.2) the sine in series and integrate with respect to a. In the obtained expansion integrals of θ are calculated using the theory of residues. Then, after passing to limit ($\beta \to 0$), we obtain

$$\zeta = \frac{qg}{2\pi\rho U^4} \sum_{m=0}^{\infty} (-1)^m \frac{[(2m+1)!!]^2}{(4m+1)!!} K^{2m-1} \Phi_{2m}(T,\gamma) \qquad (2.1)$$

$$\Phi_{2m}(T,\gamma) = \int_0^T \tau^{4m+1} R(\tau)^{-2m-3} d\tau$$

$$T \leqslant \text{const} < \infty, \; KT^2 \leqslant \text{const} < \infty \;\; (KT \leqslant \text{const} < \infty)$$

$$0 < \gamma_0 \leqslant |\gamma|, \quad \gamma_0 = \text{const}$$

We pass in $\Phi_{2m}(T,\gamma)$ to the new variable of integration R ($\operatorname{Re} R > 0$), i.e.

$$\tau = \cos\gamma + \delta_1 \sqrt{R^2 - \sin^2\gamma}, \quad \delta_1 = \operatorname{sign}(\tau - \cos\gamma)$$

Using formulas 2.110 and 2.260 (2), and 2.265 and 2.271 (4) [4] we obtain

$$\Phi_{2m}(T,\gamma) = \sum_{s=0}^{4m+1} \binom{4m+1}{s} (\cos\gamma)^{4m-s+1} I_{2m+2}^{s-1} \qquad (2.2)$$

$$I^{s-1}_{2m+2} = \sum_{p=0}^{m} \frac{B_p}{(\sin\gamma)^{2(p+1)}} \left\{ \frac{(T-\cos\gamma)^{s+1}}{[R(T)]^{2(m-p)+1}} - (-\cos\gamma)^{s+1} \right\} - \frac{B_{m+1}G_{s-1}}{(\sin\gamma)^{2(m+1)}} \quad (s \neq 1)$$

$$I^{0}_{2m+2} = \frac{\delta}{2m+1}[1 - R(T)^{-2m-1}], \quad \delta = \begin{cases} 1, & \cos\gamma \leqslant 0 \\ -1, & \cos\gamma > 0 \end{cases}$$

$$B_0 = \frac{1}{2m+1},$$

$$B_p = \frac{(s-2m)(s-2m+2)\ldots(s-2m+2p-2)}{(2m+1)(2m-1)\ldots(2m-2p+1)}(-1)^p$$

$$G_{s-1} = \sum_{q=0}^{[(s-1)/2]} \eta_q (\sin\gamma)^{2q} \{R(T)(T-\cos\gamma)^{s-2q-1} - (-\cos\gamma)^{s-2q-1}\} - \lambda\delta^{s-1} \quad (s \geqslant 1)$$

$$\eta_0 = \frac{1}{s}, \quad \eta_q = \frac{(s-1)(s-3)\ldots(s-2q+1)}{s(s-2)\ldots(s-2q)}(-1)^q$$

$$\lambda = 0, \ s = 2s_1 + 1 \ (s_1 \geqslant 0); \ \lambda = \frac{(2s_1-1)!!}{(2s_1)!!} G_{-1}, \quad s = 2s_1 \quad (s_1 > 0)$$

$$G_{-1} = \ln L; \quad L = (1+\cos\gamma)/[R(T) + |T - \cos\gamma|]$$
$$0 \leqslant T \leqslant \cos\gamma \ (\cos\gamma \leqslant 0)$$

$$L = (1+\cos\gamma)[R(T) + |T-\cos\gamma|]\,|\sin\gamma|^{-2}$$
$$0 < \cos\gamma < T$$

$$0 < |\gamma| < \pi, \quad \binom{\nu}{s} = \frac{\nu(\nu-1)\ldots(\nu-s+1)}{s!}, \quad \binom{\nu}{0} = 1$$

In the case of $|\gamma| \to \pi$ we expand the integrand in series in powers of the complex $[\sin\gamma / (\tau - \cos\gamma)]^2 < 1$, integrate with respect to τ, and obtain

$$\Phi_{2m}(T,\gamma) = \sum_{s=0}^{\infty} \binom{-m-{}^3/_2}{s} (\sin\gamma)^{2s} \chi_{m,s} \tag{2.3}$$

$$\chi_{m,s} = \sum_{p=0}^{\omega} a_p (\cos\gamma)^p T^{4m-p+1}(T-\cos\gamma)^{-\varkappa} - \eta, \quad \varkappa = 2(m+s+1)$$

$$\eta = (\varkappa+1)a_\omega \left[\sum_{q=0}^{\varkappa-1} d_q \left(\frac{T}{T-\cos\gamma}\right)^{\varkappa-q} - \ln\frac{T-\cos\gamma}{|\cos\gamma|} \right]$$

$$s \leqslant m-1 \quad (m>1), \quad \omega = 2(m-s-1)$$
$$\eta = \varkappa^{-1} a_\omega (\cos\gamma)^{\omega+1}[(T-\cos\gamma)^{-\varkappa} - |\cos\gamma|^{-\varkappa}]$$

$$s > m - 1 \quad (m \geqslant 0), \ \omega = 4m$$

$$a_0 = \frac{1}{2m - 2s - 1}$$

$$a_p = \frac{(4m+1)(4m)\dots(4m-p+2)}{(2m-2s-1)(2m-2s-2)\dots(2m-2s-p-1)}$$

$$d_q = [\varkappa(\varkappa - 1)\dots(\varkappa - q)]^{-1}$$

Formulas (2.1)-(2.3) provide the solution of the problem for the forward front of waves and indicate the wave pattern near their surge except the half-line $\gamma = 0$ and the coordinate origin.

3. We shall now analyze the rise of the fluid free surface in the zones of basic perturbations $(K \to \infty)$. Using formulas 3.937 [4] we integrate (1.2) with respect to θ, and obtain

$$\zeta = -\frac{qgK}{2\pi\rho U^4} \int_0^T \int_0^\infty a^{3/2} \exp(g\beta U^{-2}a) \sin(K\tau\sqrt{a}) J_0(KaR(\tau))\, da\, d\tau$$

We then substitute for Bessel's function its asymptotics and, taking into account formulas 9.243 (2), 9.253, 8.956, and 8.254 [4], for the estimate of the residue we obtain

$$\zeta = -Q\,[\operatorname{Im} V_0 - 11/2K^{-1}\operatorname{Re} V_1 + 9/16\pi^{1/2}K^{-3/2}V_2] + R_\zeta \qquad (3.1)$$

$$V_j = \int_0^T \psi_j(\tau)\, e^{iK\varphi(\tau)}\, d\tau \quad (j = 0, 1), \quad V_2 = \int_0^T R^{-5/2}(\tau)\, d\tau$$

$$\varphi(\tau) = \frac{\tau^2}{4R(\tau)}, \quad \psi_j(\tau) = \frac{\tau^S}{R^M(\tau)}, \quad S = 3 - 2j, \quad M = 4 - j$$

$$(j = 0, 1)$$

$$Q = \frac{qg}{2^{7/2}\pi\rho U^4}, \quad |R_\zeta| = O(K^{-3/2-\alpha}), \quad 0 \leqslant \alpha \leqslant 1/2$$

$$|\gamma| \geqslant \gamma_0 > 0, \quad \gamma_0 = \text{const}$$

The use of the Van der Korput neutralizer [5, 6] show that integral V_j $(j = 0, 1)$ depends only on contributions of the integration interval end and on fixed points τ_m $(m = 0, 1, 2)$ of the phase function $\varphi(\tau)$ of that interval where

$$\tau_0 = 0, \quad \tau_{1,2} = (3\cos\gamma \mp \sqrt{9\cos^2\gamma - 8})/2$$

$$\varphi'(\tau_m) = 0 \quad (m = 0, 1, 2)$$

$$\gamma_0 \leqslant |\gamma| \leqslant \gamma_*, \quad \cos\gamma_* = 2\sqrt{2}/3$$

Let us determine the contribution of the end point T to integrals V_j $(j = 0, 1)$. Assuming in the interval $[0, T]$ the Van der Korput neutralizer

$$\nu(\tau) = 1, \frac{d^n}{d\tau^n}\nu(\tau) = 0 \quad (n = 1, 2, \ldots), \quad 0 \leqslant \tau \leqslant \eta$$

$$\frac{d^m}{d\tau^m}\nu(\tau) = 0 \quad (m = 0, 1, \ldots), \quad T - \eta \leqslant \tau \leqslant T$$

$$\eta \leqslant T/2, \quad T - \eta < R(T)$$

then

$$V_j = \chi_j^+(0) + \chi_j(T) \quad (j = 0, 1) \tag{3.2}$$

$$\chi_j^+(0) = \int_0^{T-\eta} \nu(\tau)\psi_j(\tau)e^{iK\varphi(\tau)}\,d\tau$$

$$\chi_j(T) = \int_\eta^T [1 - \nu(\tau)]\psi_j(\tau)e^{iK\varphi(\tau)}\,d\tau$$

The contribution of point τ_0

$$\chi_0^+(0) = O(K^{-2}), \quad \chi_1^+(0) = O(K^{-1}) \tag{3.3}$$

was shown in [7] to correspond to nonwave-form deformations of the free surface. To determine the contribution of point T we apply to the integral $\chi_j(T)$ Fourier's method. We have

$$\chi_j(T) = e^{iK\varphi(T)}[R(T)]^{-j}\sum_{n=0}^{N} d_n(S, M)[KR(T)]^{-n-1} + R_\chi \tag{3.4}$$

$$0 < \gamma_0 \leqslant |\gamma|, \quad |K\varphi'(T)| \to \infty, \quad K^{-j}|R_\chi/R_\zeta| \sim \text{const}$$

$$d_n(S, M) = \Gamma\left(\frac{n+1}{B}\right)B^{-1}\exp\left(i\pi\delta\frac{n+1}{2B}\right) \times$$

$$\sum_{m=0}^{n}(m+1)a_m h_{n-m}(S, M)$$

$$h_n(S, M) = \sum_{m=0}^{n} q_m(S, M)c_{n-m}(m)$$

$$c_0(m) = a_0^m, \quad c_n(m) = \frac{1}{na_0}\sum_{k=0}^{n}(km - n + k)a_k c_{n-k}(m), \quad n \geqslant 1$$

$$q_m(S, M) = \sum_{n=0}^{\Omega}\binom{S}{S-n}\delta_1^{S-n}H^{S-n}C_{m-n}^{M/2}(\delta_1\mu)$$

$$\Omega = \min(m, S), \quad H = T/R(T), \quad \mu = (\cos\gamma - T)/R(T)$$

$$a_0 = \frac{1}{|b_0|^{1/B}}, \quad a_1 = -\frac{a_0^2}{B}\frac{b_1}{b_0}$$

$$a_n = -\frac{a_0^{n+1}}{B}\frac{b_n}{b_0} - \frac{1}{Ba_0{}^B}\sum_{m=1}^{n-1}\left\{\frac{mB-n+m}{n}a_m c_{n-m}(B) - \right.$$

$$\left.\frac{b_{n-m}}{b_0}\sum_{k=1}^{m}\frac{k(B+n+1)-(k+1)m}{m}a_k c_{m-k}(B+n-m)\right\}, \quad n \geqslant 2$$

$$b_n = p_{B+n}, \quad p_m \equiv \frac{1}{4}q_m(2,1) = \delta_1{}^m R^{1-m}(T)\frac{\varphi^{(m)}(T)}{m!}$$

where $C_m{}^\nu(\mu)$ are Gegenbauer's polynomials. In this case ($0 < T < \tau_1$ or $T \to \tau_1 - 0$, $K\varphi'(T) \to \infty$) we set in (3.4) $\delta = \delta_1 = -1$, and $B = 1$.

Formulas (3.2)-(3.4) determine the wave contribution to solution (3.1) be defining the wave development process up to the instant of time at which the ship's wake has not yet appeared ($0 < T < \tau_1$), while that process for the instants of time ($\tau_1 \leqslant T \leqslant \tau_2$; $T > \tau_2$), corresponding to the forming and formed ship's wake and ($\chi_j(T) \to 0$, $T \to \infty$) for further development and establishment of phenomena outside the ship's wake.

Let us consider the formation of the system of longitudinal waves of the ship's wake. In that case $T \to \tau_1 - 0$, so that $K\varphi'(T) \to 0$ if $\gamma_0 \leqslant |\gamma| \leqslant \gamma_1^- < \gamma_*$. The methods of Fourier and of stationary phase in its conventional form are no longer applicable to the integral $\chi_j(T)$ of formulas (3.2) owing to $K\varphi'(T) \to 0$ and $\varphi'(T) \neq 0$, respectively. In this case by transforming the phase function $\varphi(\tau)$ we separate the exponential term with its exponent tending to zero, and combine it with the amplitude function. This yields the new phase and amplitude functions $\Phi(\tau)$ and $\Psi_j(\tau; K)$

$$\Phi(\tau) = \varphi(\tau) - \varphi(T) - \varphi'(T)(\tau - T), \quad \Phi'(T) = 0, \quad \Phi''(T) \neq 0 \qquad (3.5)$$
$$\Psi_j(\tau; K) = \nu(\tau)\psi_j(\tau)\exp[iK\varphi'(T)(\tau - T)], \quad K\varphi'(T) \to 0, \quad K \to \infty$$

The integral $\chi_j(T)$ now assumes the form

$$\chi_j(T) = e^{iK\varphi(T)}\int_{\tau_1}^{T}\psi_j(\tau; K)\, e^{iK\Phi(\tau)}\, d\tau \quad (j = 0, 1)$$

to which, on the strength of properties (3.5) of function $\Phi(\tau)$, we apply the method of the stationary phase used in [7]. Denoting $\chi_j(T)|_{T\to\tau_1-0}$ by $\chi_j^-(T)|_{T\to\tau_1}$, we obtain

$$\chi_j^-(T)|_{T\to\tau_1} = e^{iK\varphi(T)}\sum_{n=0}^{N}(-1)^n\lambda_n{}^j K^{-(n+1)/2} + R_\chi \qquad (3.6)$$
$$0 < \gamma_0 \leqslant |\gamma| \leqslant \gamma_1^- < \gamma_*, \quad K\varphi'(T) \to 0, \quad K \to \infty$$
$$K^{-j}|R_\chi/R_\zeta| \sim \text{const}$$
$$|\lambda_{N+1}^j| K^{-N/2-j-1} \leqslant |R_\zeta| \leqslant |\lambda_N{}^j| K^{-(N+1)/2-j}$$

$$\lambda_n^j = \sum_{m=0}^{n} d_m(S, M)\, r_{n-m} R^{-(n+1)/2-j}(T),$$

$$r_n = \sum_{m=0}^{n} \frac{[iK\varphi'(T)\,R(T)]^m}{m!}\, c_{n-m}(m)$$

where coefficients $d_m(S, M)$, and $c_n(m)$ are defined in (3.4) where it is assumed that $\delta = -1$, $\delta_1 = 1$, and $B = 2$. Hence

$$V_j = \chi_j^+(0) + \chi_j^-(T)|_{T\to\tau_1} \quad (j = 0, 1) \tag{3.7}$$

Formulas (3.3), (3.6), and (3.7) determine the wave contribution to solution (3.1) by defining the process of generation of the system of longitudinal waves in the ship's wake.

The final stage of longitudinal wave formation in the ship's wake is defined as follows $\tau_1 \leqslant T < \tau_2$, and $T \to \tau_1 + 0$ so that $K\varphi'(T) \to 0$ when $K \to \infty$. We reduce V_j (see (3.1)) to the form

$$V_j = I_0^j + I_1^j \quad (j = 0, 1) \tag{3.8}$$

$$I_0^j = \int_0^{\tau_1} \psi_j(\tau)\, e^{iK\varphi(\tau)}\, d\tau, \quad I_1^j = \int_{\tau_1}^{T} \psi_j(\tau)\, e^{iK\varphi(\tau)}\, d\tau$$

Evidently I_0^j identical to V_j determined by formulas (3.7) if we set in the latter $T = \tau_1$. It should be noted that the limit values $\chi_j^-(T)$ in formulas (3.6) when $T \to \tau_1$ (we denote these by $\chi_j^-(\tau_1)$) are known contributions of the stationary point τ_1 (see formulas 4.5 in [7]). We have

$$I_0^j = \chi_j^+(0) + \chi_j^-(\tau_1) \quad (j = 0, 1) \tag{3.9}$$

To separate the contributions of points τ_1, and T we apply the device of the Van der Korput neutralizer to integrals I_1^j and, using the method stationary phase, as was done for $T \to \tau_1 - 0$, obtain

$$V_j = \chi_j^+(0) + \chi_j^-(\tau_1) + \chi_j^+(\tau_1) - \chi_j^+(T)|_{T\to\tau_1} \quad (j = 0, 1) \tag{3.10}$$

$$\chi_j^+(T)|_{T\to\tau_1} = e^{iK\varphi(T)} \sum_{n=0}^{N} \lambda_n^j K^{-(n+1)/2} + R_\chi$$

$$0 < \gamma_0 \leqslant |\gamma| \leqslant \gamma_1^- < \gamma_*, \quad K\varphi'(T) \to 0, \quad K \to \infty$$

$$K^{-j} |R_\chi / R_\zeta| \sim \text{const}$$

$$|\lambda_{N+1}^j| K^{-N/2-j-1} \leqslant |R_\zeta| \leqslant |\lambda_N^j| K^{-(N+1)/2-j}$$

$$\chi_j^\pm(\tau_1) = \lim \chi_j^\pm(T)|_{T\to\tau_1}$$

where $\chi_j^+(0)$ are determined by formulas (3.3) and coefficients λ_n^j by (3.6) and (3.4) where it is assumed that $\delta = -1$, $\delta_1 = 1$, and $B = 2$.

The above case relates to the development of the system of longitudinal waves in the ship's wake.

Let $\tau_1 < T < \tau_2$, then

$$V_j = \chi_j^+(0) + \chi_j^-(\tau_1) + \chi_j^+(\tau_1) + \chi_j(T) \quad (j = 0, 1) \qquad (3.11)$$
$$\gamma_0 \leqslant |\gamma| \leqslant \gamma_1^-, \quad |K\varphi'(T)| \to \infty$$

where $\chi_j^+(0)$ are determined by formulas (3.3) and $\chi_j^\pm(\tau_1)$ is the limit value of $\chi_j^\pm(T)|_{T\to\tau_1}$ (the latter are specified in (3.6) and (3.10), while $\chi_j(T)$ were determined earlier (see (3.4)) where $\delta = 1$, $\delta_1 = -1$, and $B = 1$.

Formula (3.11) determines the wave contribution in the solution (3.1) at the instant of time, when the system of longitudinal waves has been already established while the system of transverse waves has not yet emerged.

If $T \to \tau_2 \pm 0$, and $K\varphi'(T) \to 0$, the solution is derived similary to that in the case of $T \to \tau_1 \pm 0$. We have

$$V_j = \chi_j^+(0) + \chi_j^-(\tau_1) + \chi_j^+(\tau_1) + \chi_j^-(T)|_{T\to\tau_2} \quad (j = 0, 1) \qquad (3.12)$$
$$\gamma_0 \leqslant |\gamma| \leqslant \gamma_1^-, \quad K\varphi'(T) \to 0$$
$$V_j = \chi_j^+(0) + \chi_j^-(\tau_1) + \chi_j^+(\tau_1') + \chi_j^-(\tau_2) + \chi_j^+(\tau_2) - \chi_j^+(T)|_{T\to\tau_2} \quad (j = 0, 1) \qquad (3.13)$$
$$\chi_j^\pm(T)|_{T\to\tau_2} = e^{iK\varphi(T)} \sum_{n=0}^{\infty} (\pm 1)^n \lambda_n^j K^{-(n+1)/2} + R_\chi$$
$$0 < \gamma_0 \leqslant |\gamma| \leqslant \gamma_1^- < \gamma_*, \quad K\varphi'(T) \to 0, \quad K \to \infty$$
$$K^{-j} |R_\chi / R_\zeta| \sim \text{const}$$
$$|\lambda_{N+1}^j| K^{-N/2-j-1} \leqslant |R_\zeta| \leqslant |\lambda_N^j| K^{-(N+1)/2-j}$$
$$\chi_j^\pm(\tau_m) = \lim \chi_j^\pm(T)|_{T\to\tau_m} \quad (m = 1, 2)$$

where coefficients λ_n^j are defined in (3.6) for $\delta = \delta_1 = 1$, and $B = 2$.

Formulas (3.13) and (3.12) defined the formation and development of the system of transverse waves in the ship's wake.

When $T > \tau_2$ including $T \to \infty$, we have

$$V_j = \chi_j^+(0) + \chi_j^-(\tau_1) + \chi_j^+(\tau_1) + \chi_j^-(\tau_2) + \chi_j^+(\tau_2) + \chi_j(T) \qquad (3.14)$$
$$(j = 0, 1)$$
$$0 < \gamma_0 \leqslant |\gamma| \leqslant \gamma_1^- < \gamma_*, \quad |K\varphi'(T)| \to \infty, \quad K \to \infty$$

where $\chi_j(T)$ is determined as before (for $0 < T < \tau_1$) by formulas (3.4) where $\delta = 1$, $\delta_1 = -1$, and $B = 1$.

In (3.14) the values of V_j correspond to the formed systems of longitudinal and transverse waves and define their subsequent development and establishment ($\chi_j(T) \to 0$ when $T \to \infty$).

4. Let us derive the asymptotic expansion of integrals V_j $(j = 0, 1)$ determined by formulas (3.1) near the boundary of the ship's wake $(|\gamma| \to \gamma_*)$. In the indicated zone a merger of two stationary points of the phase function of integrals V_j $(\lim \tau_m = \tau_*, |\gamma| \to \gamma_*$ $(m = 1, 2))$ takes place in the approach along internal paths of region $\gamma_0 \leqslant |\gamma| \leqslant \gamma_*$ that corresponds to the ship's wake. A numerical analysis of stabilized problem carried out by Hogner appears in [3]. The theory of Euler's functions was used in [8] for the asymptotic analysis of integrals in connection with an investigation of the conversion of single stationary points to double points. Further increase of the multiplicity of stationary points would necessitate the introduction of new special functions of Airy's type [9].

The proposed here scheme makes possible the expansion of integrals by the method of stationary phase without recourse to substantial additional investigation not only in the case of increased multiplicity of internal stationary points but, also, in the case of appearance of stationarity in integration intervals, in particular when the limit of integration tends to a stationary point (single, multiple, and merging stationary points).

Let us investigate the process of formation of the ship's wake region boundary. Let $T \to \tau_* - 0$ be such that $K\{\varphi'(T), \varphi''(T)\} \to 0$, $K \to \infty$. We transform the integrals $\chi_j(T)$ defined by formulas (3.2), construct new phase $\Phi(\tau)$ and amplitude $\Psi_j(\tau; K)$ functions of the form

$$\Phi(\tau) = \varphi(T) + \varphi'(T)(\tau - T) + \frac{\varphi''(T)}{2!}(\tau - T)^2 - \varphi(\tau)$$

$$\Phi'(T) = \Phi''(T) = 0, \quad \Phi'''(T) \neq 0$$

$$\Psi_j(\tau; K) = (1 - \nu(\tau))\psi_j(\tau)\exp\left\{iK\left[\varphi'(T)(\tau - T) + \frac{\varphi''(T)}{2!}(\tau - T)^2\right]\right\}$$

and obtain

$$\chi_j(T)|_{T\to\tau_*-0} = e^{iK\varphi(T)}\int_{\eta}^{T}\Psi_j(\tau; K)e^{iK\Phi(\tau)}d\tau \quad (j = 0, 1) \tag{4.1}$$

Owing to the properties of the phase and amplitude functions of integral (4.1), we can apply to it the method of stationary phase. We substitute the variables

$$\xi^3 = \Phi(\tau), \quad \operatorname{Re}\xi > 0; \quad \tau = T + R(T)\sum_{m=0}^{\infty} a_m\xi^{m+1}$$

Further application of the method of stationary phase [7] to integrals (4.1) makes possible to represent their asymptotic expansions as

$$\chi_0^-(T)|_{T\to\tau_*} = e^{iK\varphi(T)}\sum_{n=0}^{3}\lambda_n^0 K^{-(n+1)/3} + O(K^{-5/3}) \tag{4.2}$$

$$\chi_1^-(T)|_{T\to\tau_*} = e^{iK\varphi(T)}\lambda_0^1 K^{-1/3} + O(K^{-2/3})$$

$$|\gamma| \to \gamma_*, K\{\varphi'(T), \varphi''(T)\} \to 0, \quad K \to \infty, \quad T \to \tau_* - 0$$

$$\lambda_n^j = \sum_{m=0}^{n} d_m(S, M) r_{n-m} R(T)^{-(n+1)/2-j} \quad (j = 0, 1)$$

$$r_n = \sum_{m=0}^{n} \eta_m c_{n-m}(m)$$

$$\eta_{2n+v} = \sum_{k=0}^{n} \frac{[ik\varphi''(T)]^{n-k}}{2^{n-k}(n-k)!} \frac{[ik\varphi'(T)R(T)]^{2k+v}}{(2k+v)!} \quad (v = 0, 1)$$

where the coefficients $c_n(m)$ and $d_n(S, M)$ for $\delta = \delta_1 = -1$, and $B = 3$ are determined in (3.4).

Taking into account formulas (3.2), (3.3), and (4.2) we obtain

$$V_j = \chi_j \;\, (0) + \chi_j^-(T)\,|_{T\to\tau_*} \; (j = 0, 1) \tag{4.3}$$
$$|\gamma| \to \gamma_*, \quad K\{\varphi'(T), \varphi''(T)\} \to 0, \quad K \to \infty$$

The asymptotic expansions (4.3) define the process of formation of the ship's wake boundary.

Let us consider the case of $|\gamma| \to \gamma_*$ and $T \to \tau_* + 0$ that corresponds to the development of the ship's wake boundary. The integrals $\chi_j(T)$ in (3.2) are treated similarly to those in the case of $T \to \tau_1 + 0$. With allowance for (3.2) and (3.3) we obtain for V_j asymptotics of the form

$$V_j = \chi_j^+(0) + \chi_j^-(\tau_*) + \chi_j^+(\tau_*) - \chi_j^+(T)\,|_{T\to\tau_*} \quad (j = 0, 1) \tag{4.4}$$
$$\chi_0^+(T)\,|_{T\to\tau_*} = e^{iK\varphi(T)} \sum_{n=0}^{3} \lambda_n^{\circ} K^{-(n+1)/3} + O(K^{-5/3})$$
$$\chi_1^+(T)\,|_{T\to\tau_*} = e^{iK\varphi(T)} \lambda_0^1 K^{-1/3} + O(K^{-2/3})$$
$$|\gamma| \to \gamma_*, \quad K\{\varphi'(T), \quad \varphi''(T)\} \to 0, \quad K \to \infty, \quad T \to \tau_* + 0$$
$$\chi_j^{\pm}(\tau_*) = \lim \chi_j^{\pm}(T)\,|_{T\to\tau_*}$$

where the coefficients $\lambda_n^j\,|_{T\to\tau_*+0}$ are determined in (4.2) for $\delta = -1, \delta = 1$, and $B = 3$.

Formulas (4.4) define the development process of the ship's wake boundaries. Asymptotic expansions (4.3) and (4.4) are the same when $T = \tau_*$.

Asymptotics of integrals V_j $(j = 0, 1)$, are similarly derived when $T > \tau_*$, and $|\gamma| \to \gamma_*$ so that $K\{\varphi'(\tau_*)$, and $\varphi''(\tau_*)\} \to 0$ when $K \to \infty$, which corresponds to the established ship's wake and includes the process of its establishment. We have

$$V_j = \chi_j^+(0) + \chi_j^-(\tau_*) + \chi_j^+(\tau_*) + \chi_j(T) \quad (j = 0, 1) \tag{4.5}$$

where $\chi_j^+(0)$, and $\chi_j(T)$ are the known contributions of the end points of the integration interval (3.3) and (3.4), and functions $\chi_j^{\pm}(\tau_*)$ are determined by formulas (4.2) and (4.4) for $T = \tau_*$.

5. The complete wave pattern determined by V_j $(j = 0, 1)$ is superposed on the free surface deflection that is defined by the integral in formula (3.1). When $0 < |\gamma| < \pi$ the substitution $\xi = (\tau - \cos\gamma) / |\sin\gamma|$ reduces integral V_2 to elliptic integrals (3.185(5) [4])

$$V_2 = \frac{\sqrt{2}}{|\sin\gamma|^{3/2}} \left\{ \delta_1 \left[2E\left(\alpha_1, \frac{1}{\sqrt{2}}\right) - F\left(\alpha_1, \frac{1}{\sqrt{2}}\right) \right] + \delta_2 \left[2E\left(\alpha_2, \frac{1}{\sqrt{2}}\right) - F\left(\alpha_2, \frac{1}{\sqrt{2}}\right) \right] \right\} \quad (5.1)$$

$$\alpha_1 = \arccos\sqrt{|\sin\gamma| / R(T)}, \quad \alpha_2 = \arccos\sqrt{|\sin\gamma|}, \quad 0 < |\gamma| < \pi$$

$$\delta_1 = 1 \quad \delta_2 = -1, \quad \cos\gamma \leqslant 0$$

$$\delta_1 = -1, \quad \delta_2 = 1, \quad \cos\gamma > 0, \quad T \leqslant \cos\gamma$$

$$\delta_1 = 1, \quad \delta_2 = 1, \quad \cos\gamma > 0, \quad T > \cos\gamma$$

Let $|\gamma| \to \pi$. We represent $R(\tau)$ as follows:

$$R(\tau) = (\tau - \cos\gamma)(1 + \varepsilon)^{1/2}, \quad \varepsilon = [\sin\gamma / (\tau - \cos\gamma)]^2$$

Expanding the integrand in series for $\varepsilon < 1$, and integrating we obtain

$$V_2 = \sum_{n=0}^{\infty} C_n (\sin\gamma)^{2n} [|\cos\gamma|^{-2n-3/2} - (T - \cos\gamma)^{-2n-3/2}] \quad (5.2)$$

$$C_n = (-1)^n \frac{1\cdot 5\cdot 9\ldots(4n+1)}{n!\,(4n+3)} \cdot 2^{1-2n}$$

$$|\gamma| \to \pi, \quad \frac{3}{4}\pi < |\gamma| < \frac{5}{4}\pi$$

Deflection of the fluid free surface is determined by formulas (5.1) and (5.2) including its development and establishment.

It should be noted that the principal terms of the asymptotics of solution outside the ship's wake region are the same as in earlier solutions [1]. A similar agreement between principal terms of asymptotic formulas for the systems of longitudinal and transverse waves and, also, for the ship's wake boundaries is observed in the stabilized case [3, 7].

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REFERENCES

1. Stoker, J., Water Waves, Wiley, New York, 1957.

2. Cherkesov, L. V., Unstable Waves. Kiev, "Naukova Dumka", 1970.

3. Sretenskii, L. N., Theory of Wave Motion of Fluid. Moscow-Leningrad, ONTI, 1936.

4. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series, and Products. Moscow, "Nauka", 1971.

5. Erdelyi, A. Asymptotic Expansions, Dover, New York, 1956.

6. F o c k e, J. Asymptotische Entwiklungen mittels der Methode der Stationäran Phase. Berli, Verhandl. Sächsisch. Akad. Wiss. mathem-naturwiss. Vol. 101, No. 3, 1954.

7. Z o l o t a r e v, A. A., On the problem of Ships Waves. Collect'n: Hydrophysical Marine Investigations, No. 1 (64). Sevastopol. Izd. Akad . Nauk Ukr. SSSR, 1974.

8. C h e s t e r, C., Friedman, B., and Ursell, F., An extension of the method of steepest descents. Proc. Cambridge Philos. Soc., Vol. 53, pt. 3, 1957.

9. B l e s t e i n, N., Uniform asymptotic expansions of integrals with many nearly stationary points and algebraic singularities. J. Math. and Mech. Vol. 17, No. 6, 1967.

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